

## ON RIGIDITY OF GRAUERT TUBES

SU-JEN KAN

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ABSTRACT.

Given a real-analytic Riemannian manifold  $M$  there exists a canonical complex structure on part of its tangent bundle which turns leaves of the Riemannian foliation on  $TM$  into holomorphic curves. A *Grauert tube* over  $M$  of radius  $r$ , denoted as  $T^r M$ , is the collection of tangent vectors of  $M$  of length less than  $r$  equipped with this canonical complex structure. We say the Grauert tube  $T^r M$  is *rigid* if  $\text{Aut}(T^r M)$  is coming from  $\text{Isom}(M)$ .

In this article, we prove the rigidity for Grauert tubes over quasi-homogeneous Riemannian manifolds. A Riemannian manifold  $(M, g)$  is quasi-homogeneous if the quotient space  $M/\text{Isom}_0(M)$  is compact. This category has included compact Riemannian manifolds, homogeneous Riemannian manifolds, co-compact Riemannian manifolds whose isometry groups have dimensions  $\geq 1$ , and products of the above spaces.

**1. Introduction.**

The *adapted complex structure* is the unique complex structure on (part of) the tangent bundle of a real-analytic Riemannian manifold  $(M, g)$  which turns leaves of the Riemannian foliation on  $TM$  into holomorphic curves. It was proved by Guillemin-Stenzel and independently by Lempert-Szöke that such an adapted complex structure exists in a neighborhood of  $M$  in  $TM$  by solving a complex homogeneous Monge-Ampère equation. We make a remark here that the Monge-Ampère solutions  $\rho_{GS}$  in [G-L],  $\rho_{LS}$  in [L-S] and  $\rho_K$  in [K1] all have different scalings. The relation among these three are:  $\rho_{GS} = 2\rho_K = 4\rho_{LS}$ . The  $\rho_{LS}$  is exactly the length square function. Therefore, we will adapt the scaling that used in [L-S]; the  $\rho$  we consider in this article is the Monge-Ampère solution with the initial condition  $\rho_{i\bar{j}}|_M = \frac{1}{2}g_{ij}$  and  $\rho(x, v) = |v|_g^2$  for any tangent vector  $v \in T_x M$ . The set of tangent vectors of length less than  $r$  equipped with the adapted complex structure is called a *Grauert tube* of radius  $r$  centered at  $M$ . It is denoted as  $T^r M$ . In terms of the Monge-Ampère solution,  $T^r M = \{\rho^{-1}[0, r^2)\}$ .

Associated to each  $M$  there exists a maximal possible radius  $r_{max}(M) \geq 0$  such that  $T^r M$  exists for any  $r \leq r_{max}(M)$ . Since the adapted complex structure is locally defined, the maximal possible radius  $r_{max}(M) > 0$  for compact  $M$ . Nevertheless,  $r_{max}(M)$  might very well be zero for non-compact  $M$ .

Since the adapted complex structure is constructed canonically associated to the Riemannian metric  $g$  of the manifold  $M$ , the differentials of the isometries of  $(M, g)$  are natural automorphisms of  $T^r M$ . On the other hand, it is interesting to see whether every automorphism of  $T^r M$  comes from this way or not. When the answer is affirmative, we say the Grauert tube is *rigid*.

With respect to the adapted complex structure, the length square function  $\rho$  is strictly plurisubharmonic. When the center  $M$  is compact the Grauert tube  $T^r M$  is Stein since it is exhausted by the strictly plurisubharmonic function  $\rho$ . Furthermore, if the radius  $r$  is less than the maximal possible radius,  $T^r M$  is then a bounded domain with smooth strictly pseudoconvex boundary in a Stein manifold. In this case, most good properties of bounded strictly pseudoconvex domains are inherited and the automorphism group of  $T^r M$  is a compact Lie group. Base on these, Burns and Hind ([B-H]) are able to prove the rigidity for Grauert tubes over compact real-analytic Riemannian manifolds.

For non-compact  $M$ , the length square function  $\rho$  is no longer an exhaustion. This has made the situation complicated to deal with. In fact, nothing particular is known, not even to the existence of a Grauert tube over non-compact  $M$ . When the Grauert tubes exist, most of the good properties in the compact case are lacking here since the length square function  $\rho$  no longer exhausts the tube.

In [K2], using the homogeneity of the central manifolds, the author was able to prove the rigidity of Grauert tubes over homogeneous Riemannian manifolds. The author also proved a weak rigidity for general Grauert tubes which says that if  $r_{max}(M) > 0$  and  $T^r M$  is not covered by the ball, then  $Aut_0(T^r M) \cong Isom_0(M)$  for any  $r < r_{max}(M)$ .

It does not seem clear to us what kind of geometric properties possessed by the real-analytic Riemannian manifold  $(M, g)$  will guarantee the existence the Grauert tube structure, i.e., will enforce  $r_{max}(M) > 0$ . To the author's knowledge, there are only four categories that we know:  $M$  is compact, is homogeneous, is co-compact or is the product of the above cases. The rigidity of Grauert tubes over the first two cases have been clarified in [B-H] and [K2], respectively. The motivation of this article is to examine the rigidity of Grauert tubes over the third and the fourth

cases.

In an attempt to tackle this problem, we formulate the situation as below. The objects we are interested in are real-analytic Riemannian manifolds  $M$  possesses the property that every point in  $M$  could be sent to a compact subset of  $M$  through some isometry. First of all, the above four cases(compact, homogeneous, co-compact and products of them) are all included in this family. Secondly, this is a kind of generalized homogeneous spaces; lots of the technique in the homogeneous case could be transplanted here without much difficulty. However, to prove the rigidity, we need a reasonable large isometry group. So we defined a Riemannian manifold  $M$  is quasi-homogeneous if  $Isom_0(M) \cdot K = M$  for some compact subset  $K$  of  $M$ . This would immediately implies that  $\dim Isom_0(M) \geq 1$  for non-compact  $M$ . The main result of this article is Theorem 4.5 which proved the rigidity of Grauert tubes over quasi-homogeneous Riemannian manifolds.

**Theorem 4.5.** *Let  $M$  be a quasi-homogeneous manifold. Let  $T^r M$  be the Grauert tube of radius  $r < r_{max}(M)$  which is not covered by the ball. Then  $Isom(M) \cong Aut(T^r M)$  and  $T^r M$  has a unique center.*

The organization of this article is as the following. In §2 we define quasi-homogeneity and derive some basic properties of quasi-homogeneous manifolds. We prove that a Grauert tube over quasi-homogeneous manifold is complete hyperbolic in §3. The §4 is devoted to the proof of the rigidity of  $T^r M$  for quasi-homogeneous  $M$ .

The author would like to thank Professor Kang-Tae Kim for bringing her attention to the case  $M/Isom(M)$  is compact.

## §2 Quasi-homogeneous manifolds.

Recall that a Riemannian manifold  $M$  is called homogeneous if for any two points  $x, y \in M$ , there exists a  $g \in Isom(M)$  such that  $g \cdot x = y$ . This transitivity property has implied that the isometry group of  $M$  is very large, it is at least as large as  $M$ .

Motivated by the homogeneous case, the real-analytic Riemannian manifolds we consider through out this article are Riemannian manifolds possess certain kind of transitivity and have reasonably large isometry groups.

**Definition.** A real-analytic Riemannian manifold  $M$  is *quasi-homogeneous* if the quotient space  $M/Isom_0(M)$  is compact.

The condition is equivalent to that there exists a compact subset  $K$  of  $M$  such that for any  $m \in M$  there associated a  $g \in Isom_0(M)$  with  $g \cdot m \in K$  or equivalently,

$Isom_0(M) \cdot K = M$ . We may further assume that both  $M$  and  $K$  are connected and that for any  $x \neq y, x, y \in \overset{\circ}{K}$ , there is no  $g \in Isom_0(M)$  such that  $g \cdot x = y$ .

The reason we ask for  $Isom_0(M)$  instead of  $Isom(M)$  is that the criteria would force the isometry group of  $M$  to be fairly large for non-compact manifold  $M$ , say  $\dim Isom(M) \geq 1$ . In fact, if a non-compact connected manifold  $M$  has a connected compact quotient  $M/Isom(M)$  then either  $\dim Isom(M) \geq 1$  which would automatically imply that  $M/Isom_0(M)$  is compact or  $M$  is a non-compact co-compact manifold with a discrete isometry group.

This category actually has contained most of cases that we are sure about the existence of Grauert tubes, examples of such  $M$  are: compact manifolds, homogeneous manifolds, co-compact manifolds whose isometry group have dimensions  $\geq 1$ , and products of the above manifolds. The only thing out of control are non-compact co-compact manifolds with discrete isometry groups. We examine some of the fundamental properties of quasi-homogeneous spaces.

**Remark.** *The properties proved in this and the next sections hold for any Riemannian manifold  $M$  which possesses the property that there exists a compact subset  $K \subset M$  with the property that  $Isom(M) \cdot K = M$ . That is, they also work for non-compact co-compact manifolds with discrete isometry groups. However, for the rigidity proof in the last section of this article, we do need some dimension criteria on the isometry group.*

**Lemma 2.1.** *If  $(M, g)$  is a quasi-homogeneous manifold, then  $(M, g)$  is complete.*

*Proof.* To prove  $(M, g)$  is complete is equivalent to show that every geodesic could be extended to a geodesic defined on the whole real line.

Let  $K$  be a compact subset of  $M$  such that every point of  $M$  could be sent to  $K$  by an isometry  $h \in Isom_0(M)$ . Let  $s$  be a positive number such that the exponential map is defined on the ball  $B(p, s)$  for all  $p \in K$ . By hypothesis, any point  $m$  in  $M$  could be sent to a point in  $K$  by an isometry. Therefore, for any  $m \in M$ , the exponential map  $\exp_m$  is defined on  $B(m, s)$ .

Let  $\eta(t)$  be a geodesic with  $\eta(0) = m$  and  $I = (-a, b)$  be its maximal interval of definition. Then both  $a$  and  $b$  are greater than  $s$ . Suppose  $b < \infty$ , we set the point  $p = \eta(b - \frac{s}{2})$ . Let  $\gamma$  be the geodesic with  $\gamma(0) = p$  and  $\gamma'(0) = \eta'(b - \frac{s}{2})$ . Then this  $\gamma$  extends  $\eta$  and is at least defined on the interval  $(-s, s)$ . That is,  $\eta$  is defined on  $(b - \frac{3s}{2}, b + \frac{s}{2})$ , a contradiction. Similarly, we may show that  $a = \infty$ .  $\square$

**Lemma 2.2.** *If  $M$  is a quasi-homogeneous manifold. Then  $r_{\max}(M) > 0$ .*

*Proof.* Let  $K \subset M$  be the compact set described as before. As the adapted complex structure is a local object, there exists a  $r > 0$  such that the adapted complex structure is defined on  $T_p^r M := \{(p, v) : v \in T_p M, |v| < r\}$  for all  $p \in K$ . Since any point  $m \in M$  could be sent to a point in  $K$  by an isometry (hence an automorphism), the adapted complex structure is well-defined on  $T_m^r M$  for all  $m \in M$ . The lemma is claimed.  $\square$

**Lemma 2.3.** *Let  $(M, g)$  be a non-compact quasi-homogeneous manifold and let  $K \subset M$  be a compact subset described as above. Then for every  $p \in K$ , there exist infinitely many  $m \in M$  and  $h_m \in \text{Isom}_0(M)$  such that  $h_m(m) = p$ .*

*Proof.* It is clear that  $\text{Isom}_0(M)$  is non-compact. Since  $K$  is compact in a complete Riemannian manifold, it is bounded. We may pick a large geodesic ball  $B := B(p, s)$  centered at  $p$  of radius  $s$  containing  $K$ . Given  $q_1 \in M - K$ , there exists  $h_1 \in \text{Isom}_0(M)$  such that  $h_1(q_1) \in K$ . We may assume that  $h_1(q_1) = p$ . If not, there exists a large set  $L$  outside of  $K$  such that  $h_1(L) \supset K$ , then we are able to find some  $\hat{q} \in L$  such that  $h_1(\hat{q}) = p$ .

Denote  $h_1^{-1}(K) := \hat{K}_1 \subset \hat{B}_1 := h_1^{-1}(B)$ , a geodesic ball of radius  $s$  centered at  $q_1$ . Take  $q_2 \in M - \hat{B}_1$  such that  $d(q_1, q_2) > 2s$  and such that there exists a  $h_2 \in \text{Isom}_0(M)$  sending  $q_2$  to  $p$  where  $d$  denotes the distance function induced by the Riemannian metric  $g$ . Denoting  $h_2^{-1}(K) := \hat{K}_2 \subset \hat{B}_2 := h_2^{-1}(B)$ , then  $\hat{B}_2 \cap \hat{B}_1 = \emptyset$ .

As  $M$  is complete, we are able to continue the process and find infinitely many different  $q_j \in M$  and  $h_j \in \text{Isom}_0(M)$  such that  $h_j(q_j) = p, \forall j$ .  $\square$

### §3 Complete hyperbolicity of $T^r M$ .

The Kobayashi pseudo-metric is defined in any complex manifold. When it is a metric, we call such a complex manifold a hyperbolic manifold. It was shown by Sibony in [S] that a complex manifold admitting a bounded strictly plurisubharmonic function is hyperbolic. In a Grauert tube of finite radius, the length square function  $\rho$  is strictly plurisubharmonic and bounded in the whole tube. Therefore, every Grauert tube of finite radius is hyperbolic. On the other hand, a Grauert tube of infinite radius, i.e., the adapted complex structure is defined on the whole tangent bundle of  $M$ , is never hyperbolic.

**Proposition 3.1.** *Let  $TM$  be a Grauert tube of infinite radius. Then  $TM$  can not be hyperbolic.*

*Proof.* Following the definition of the adapted complex structure, for a given arc-length parametrized geodesic  $\gamma$  in  $M$  there exists a holomorphic mapping

$$f : \mathbb{C} \rightarrow TM, \quad f(\sigma + i\tau) = \tau\gamma'(\sigma).$$

Since the Kobayashi metric in  $\mathbb{C}$  is trivial, by Picard's theorem, the mapping  $f$  has to be constant if  $TM$  is hyperbolic. A contradiction.  $\square$

In general, it is hard to see whether a hyperbolic manifold is complete or not. If  $M$  is compact and the radius  $r < r_{max}(M)$  then the Grauert tube  $T^r M$  is complete hyperbolic since it is a bounded domain in the Stein manifold  $T^{r_{max}} M$  with smooth strictly pseudoconvex boundary. The same holds for co-compact  $M$  since  $T^r(M/\Gamma) = T^r M/\Gamma$  for  $\Gamma < Isom(M)$  and the Grauert tube  $T^r M$  is the covering of a complete hyperbolic manifold. In [K2], we also proved that  $T^r M$  is complete hyperbolic when  $M$  is homogeneous and  $r < r_{max}(M)$ .

In [F-S] Fornaess and Sibony has proved a sufficient condition for a hyperbolic manifold to be complete hyperbolic. They show that a hyperbolic manifold  $\Omega$  is complete hyperbolic if the quotient  $\Omega/Aut(\Omega)$  is compact. Inspired from Fornaess-Sibony's work, we prove in this section that the Grauert tube  $T^r M$  is complete hyperbolic if  $M$  is quasi-homogeneous and  $r < r_{max}(M)$ .

Let  $d_K$  be the Kobayashi metric of the hyperbolic manifold  $T^r M$  and  $\hat{d}_K$  be the restriction of  $d_K$  to  $M$ . That is, the metric  $\hat{d}_K$  is defined as

$$\hat{d}_K(p, q) := d_K(p, q), \forall p, q \in M.$$

From the construction of Grauert tubes, the  $Isom(M)$  is naturally included in the  $Aut(T^r M)$ . Therefore any  $h \in Isom(M)$  is an isometry for the metric  $\hat{d}_K$ .

Recall that a metric space is complete if every Cauchy sequence converges. Since the Kobayashi metric  $d_K$  is an inner metric (i.e., one that comes from the arc length), the Hopf-Rinow-Myers theorem shows that the completeness is equivalent to that every finite ball is relatively compact. We will prove the completeness of  $d_K$  by first showing that  $(M, \hat{d}_K)$  is complete, i.e., every Cauchy sequence converges.

**Lemma 3.2.**  *$\hat{d}_K$  is a complete metric in  $M$ .*

*Proof.* We would like to show that every Cauchy sequence in  $(M, \hat{d}_K)$  converges. Since  $M$  is quasi-homogeneous, there is a compact set  $K \subset M$  such that for any

$p \in M$  there exists a  $h \in Isom_0(M)$  with  $h(p) \in K$ . Notice that by the definition of a Riemannian manifold, the natural topology in  $M$  and the topology induced by the metric  $g$  are the same.

Since  $K$  is compact in the complete Riemannian manifold  $(M, g)$ ,  $K$  is bounded and closed. We may find a ball  $B_g(x_0, R)$ , centered at  $x_0 \in M$  of radius  $R$  with respect to the  $g$ -metric, in  $M$  such that  $K \subset B_g(x_0, R)$ . Let  $F := B_g(x_0, 2R)$ . Since  $d_K$  is continuous, the metric  $\hat{d}_K$  is continuous as well. Therefore, there exists an  $\epsilon > 0$  such that  $\hat{d}_K(p, q) \geq \epsilon$  for any  $p \in K, q \notin \bar{F}$ .

Let  $\{p_j\} \subset M$  be a Cauchy sequence in  $(M, \hat{d}_K)$ . That is, given  $\epsilon > 0$  there exists a large  $m$  such that  $\hat{d}_K(p_k, p_l) < \epsilon, \forall k, l \geq m$ . Let  $h \in Isom_0(M)$  such that  $h(p_m) \in K$ . Then

$$\hat{d}_K(h(p_j), h(p_m)) = d_K(h(p_j), h(p_m)) = d_K(p_j, p_m) = \hat{d}_K(p_j, p_m) < \epsilon, \forall j > m.$$

The Cauchy sequence  $\{h(p_j) | j = m, \dots, \infty\}$  lies in the compact set  $\bar{F}$ . It converges to a point  $\hat{p} \in \bar{F}$  and hence the Cauchy sequence  $\{p_j\}$  converges to the point  $h^{-1}(\hat{p}) \in M$ .  $\square$

### Theorem 3.3.

*Let  $M$  be a quasi-homogeneous manifold. Then for any  $r < r_{max}(M)$ ,  $T^r M$  is complete hyperbolic.*

*Proof.* Fix  $z = (p_1, v_1) \in T^r M$ . We compute the Kobayashi distance from  $z$  to the point  $w = (p_2, v_2)$ . Let  $h_1 \in Isom_0(M)$  such that  $h_1(p_2) \in K$ , then there exists a constant  $L_1(v_2)$  such that

$$(3.1) \quad d_K((p_2, 0), (p_2, v_2)) = d_K((h_1(p_2), 0), (h_1(p_2), h_{1*}v_2)) \leq L_1(v_2).$$

Now,

$$\begin{aligned} d_K(z, w) &\geq d_K((p_1, 0), (p_2, v_2)) - d_K((p_1, 0), (p_1, v_1)) \\ (3.2) \quad &\geq d_K((p_1, 0), (p_2, 0)) - d_K((p_2, 0), (p_2, v_2)) - d_K((p_1, 0), (p_1, v_1)) \\ &= \hat{d}_K(p_1, p_2) - d_K((p_2, 0), (p_2, v_2)) - d_K((p_1, 0), (p_1, v_1)). \end{aligned}$$

We claim the complete hyperbolicity by showing that for any radius  $R$  the bounded ball  $B_K(z, R)$  is relatively compact in  $T^r M$ .

Since  $T^r M$  is a domain in the complex manifold  $T^{r_{max}} M$  and every point  $(x, v) \in T_x M, |v| = r$  is a smooth strictly pseudoconvex boundary point of  $T^r M$ . By Lemma

2.2 of [K2], if  $w = (p_2, v_2) \in B_K(z, R)$  then there exists  $\delta > 0$  such that  $|v_2| < r - \delta$ . By equations (3.1) and (3.2),

$$\begin{aligned} \hat{d}_K(p_1, p_2) &\leq d_K(z, w) + d_K((p_2, 0), (p_2, v_2)) + d_K((p_1, 0), (p_1, v_1)) \\ &< R + L_1(v_2) + L_1(v_1) < L \end{aligned}$$

for some positive  $L$ . Thus, by Lemma 3.2,  $p_2$  lies in some bounded set in  $M$ . Thus  $B_K(z, R)$  is relatively compact in  $T^r M$  and the Kobayashi metric is complete.  $\square$

#### 4. The rigidity of Grauert tubes.

For compact  $X$ , Burns-Hind [B-H] have proved that the isometry group  $Isom(X)$  of  $X$  is isomorphic to the automorphism group  $Aut(T^r X)$  of the Grauert tube for any radius  $r \leq r_{max}(X)$ .

For the non-compact cases, the best rigidity results so far are in [K2]. It shows that the rigidity holds for any Grauert tube over a homogeneous Riemannian manifold of  $r < r_{max}$ ; it also claims that the identity component of the automorphism group of the Grauert tube is isomorphic to the identity component of the isometry group of the center manifold for Grauert tubes over general real-analytic Riemannian manifolds of  $r < r_{max}$ . The only exception for the above two results occurs when the Grauert tube is covered by the ball. It was proved by the author in [K1] that  $T^r X$  is biholomorphic to  $B^n \subset \mathbb{C}^n$  if and only if  $X$  is the real hyperbolic space  $\mathbb{H}^n$  of curvature  $-1$  and the radius  $r = \frac{\pi}{4}$ . Apparently, the automorphism group of  $B^n$  is much larger than the isometry group of  $\mathbb{H}^n$ . We also remark here that the restriction  $r < r_{max}$  is necessary as the rigidity fails for Grauert tubes over non-compact symmetric spaces of rank-one of maximal radius shown in [B-H-H].

Let  $\tilde{M}$  be the universal covering of  $M$ . We have proved in Lemma 4.2 of [K2] that if there is a unique Grauert tube representation for the complex manifold  $T^r \tilde{M}$  then there is a unique Grauert tube representation for  $T^r M$ . Denote  $I = Isom(M)$  and  $G = Aut(T^r M)$ ;  $I_0$  and  $G_0$  the corresponding identity components. We have shown in Theorem 6.4 of [K2] that  $G_0 = I_0$  provided that  $T^r M$  is not covered by the ball. We may assume from now on that  $M$  is a simply-connected quasi-homogeneous manifold,  $r < r_{max}(M)$  and that  $T^r M$  is not covered by the ball. Since  $M$  is a connected quasi-homogeneous Riemannian manifold, there exists a connected compact subset  $K \subset M$  such that  $I_0 \cdot K = M$  and for any  $x \neq y, x, y \in \overset{\circ}{K}$ , there is no  $g \in Isom_0(M)$  such that  $g \cdot x = y$ .



For any given  $f \in G$ , the set  $N = f(M)$  equipped with the push-forward metric coming from  $M$  is another center of the Grauert tube  $T^r M$ . It follows from the Theorem 6.4 of [K2] that  $Isom_0(N) = G_0 = Isom_0(M)$ . Therefore,  $I_0 \cdot N = N$ .

Let's denote the projection as

$$\pi : T^r M \rightarrow M, \quad \pi(p, v) = p.$$

It is clear that  $\pi$  is  $I_0$ -invariant since for any  $(x, v) \in T^r M$  and  $g \in I_0$ , we have

$$(4.1) \quad \pi(g \cdot (x, v)) = \pi(g \cdot x, g_* v) = g \cdot x = g \cdot \pi(x, v).$$

The Riemannian metric  $g$  of  $M$  determines the Levi-Civita connection on  $TM$  which splits the tangent space  $T_z(TM)$ ,  $z \in TM$ , into vertical and horizontal spaces. The connection map is  $K : T_z(TM) \rightarrow T_z(T_{\pi(z)}M)$ . A vector  $\zeta \in T_z(TM)$  is horizontal if  $K\zeta = 0$ ; is vertical if  $\pi_*\zeta = 0$ . Let's make some preliminary observation on the projection of  $N \subset T^r M$ .

**Lemma 4.1.**  $\dim(\pi(N)) = \dim M = n$ .

*Proof.* Let  $U_z \subset N$  be a small neighborhood of  $z \in N$ . We would like to show that its projection  $\pi(U_z)$  has dimension  $n$ . Suppose not,  $\dim \pi(U_z) = \mu < n$ . Let  $\dim I_0 = l$ ,  $\dim K = k$ . Then the tangent space  $T_z(U_z)$  is spanned by  $\mu$  horizontal vectors and  $n - \mu$  vertical vectors  $\{\eta_1, \dots, \eta_{n-\mu}\}$ . As  $N$  is  $I_0$  invariant, and  $I_0$  moves vertical vectors to vertical vectors. By the quasi-homogeneity of  $M$ , the dimension of  $N$  would be  $n - \mu + l + \mu$  if  $\mu < k$ ; would be  $n - \mu + n$  if  $\mu \geq k$ . In either case, the dimension of  $N$  would be greater  $n$  since  $l \geq 1$  and  $n - \mu > 0$ . A contradiction. Therefore,  $\mu = n$ .  $\square$

We would like to show that  $N$  actually crosses through each fiber  $T_p^r M := \{(p, v) : v \in T_p M, |v| < r\}$ ,  $p \in M$ . That is,  $\pi(N) = M$ .

**Lemma 4.2.**  $N \cap T_p^r M \neq \emptyset$  for any  $p \in M$ .

*Proof.* It is clear from the above lemma that  $\pi(N)$  and  $M$  have the same dimension and  $\pi(N)$  is connected since  $\pi$  is a continuous map. It is therefore sufficient to claim that  $\pi(N)$  is both open and closed. The connectedness of  $M$  would immediately implies that  $\pi(N) = M$ .

For any  $p \in \pi(N)$ , there exists a  $z \in N$  such that  $\pi(z) = p$ . Since  $N$  is a submanifold, every point is an interior point and there is a neighborhood  $U_z$  of  $z$  in

$N$ . By the proof of Lemma 4.1,  $\pi(U_z)$  has dimension  $n$  and hence is a neighborhood of  $p$  in  $\pi(N)$ . Therefore,  $\pi(N)$  is open in  $M$ .

For the closeness. We first observe that there exists a real number  $0 < l < r$  such that every  $(x, v) \in N$  has  $|v| < l$ . This comes from the fact that  $I_0 \cdot K = M$  and  $N = f(M) = f(I_0 \cdot K) = I_0 \cdot f(K)$ . Let  $\{x_j\}$  be a sequence in  $\pi(N)$  with  $\lim_{j \rightarrow \infty} x_j = x \in M$ . Then for give  $\epsilon > 0$ , there exists an  $L > 0$  such that  $x_j \in B(x_L, \epsilon), \forall j > L$ . With respect to each  $x_j$ , there associates a point  $(x_j, v_j) \in N$ . Thus  $\{(x_j, v_j) : j > L\}$  is a sequence in the compact set  $N \cap \overline{T^l B(x_L, \epsilon)}$  and thus has a limit point  $(x, v) \in N \cap \overline{T^l B(x_L, \epsilon)}$ . Thus  $x \in \pi(N)$  and  $\pi(N)$  is closed.  $\square$

The Lie group  $I$  is a subgroup of  $G$  and  $I_0 = G_0$ . We consider the group  $G/I_0$  and examine the index of this coset space.

**Proposition 4.3.** *The index of  $I_0$  in  $G$  is finite.*

*Proof.* Let  $\{g_j I_0\}, g_j \in G$ , be a sequence in the coset space  $G/I_0$ . By Lemma 4.2,  $g_j(M)$  has non-empty intersection with any fiber  $T_p^r M, p \in M$ . Fix  $p \in K$  and take a point

$$(4.2) \quad q_j \in g_j(M) \cap T_p^r M.$$

Since  $g_j^{-1}(q_j) \in M$  there exists a  $h_j^{-1} \in I_0$  such that

$$(4.3) \quad h_j^{-1} \cdot g_j^{-1}(q_j) := a_j \in K, \forall j.$$

Let  $f_j = g_j \cdot h_j \in G$  then

$$(4.4) \quad f_j(a_j) = q_j \in T_p^r M.$$

By the complete hyperbolicity proved in §3,  $T^r M$  is a taut manifold which says that we can extract a subsequence (we still call it  $\{f_j\}$ ) that either converges uniformly on compact subsets of  $T^r M$  or diverges compactly, i.e., for any compact subsets  $D_1, D_2$  of  $T^r M$  there exists a large  $\mathcal{N}$  such that  $f_j(D_1) \cap D_2 = \emptyset, \forall j > \mathcal{N}$ .

Let

$$\text{diam}(K) := \sup\{d_K(p, q) : p, q \in K\} = \mathcal{R} < \infty$$

denote the diameter of the compact set  $K$  with respect to the Kobayashi metric  $d_K$  in  $T^r M$ . Then  $K \subset B_K(a_j, \mathcal{R}), \forall a_j \in K$ . As each  $f_j$  is an isometry of the Kobayashi metric. We have

$$f_j(K) \subset f_j(B_K(a_j, \mathcal{R})) = B_K(f_j(a_j), \mathcal{R}) = B_K(q_j, \mathcal{R}), \forall j.$$

Since  $q_j \in T_p^r M$ , it is therefore possible to find a compact set  $K' \subset M$  such that

$$(4.4) \quad f_j(K) \subset T_{K'}^r M, \forall j,$$

where  $T_{K'}^r M = \cup_{p \in K'} T_p^r M$ . Let  $\Omega_k := T_{K'}^{r - \frac{1}{k}}(M)$ . Suppose the sequence  $\{f_j\}$  diverges compactly. Then for any  $k \in \mathbb{N}$  there exists an  $\mathcal{N}_k$  such that  $f_j(K) \cap \bar{\Omega}_k = \emptyset$  for all  $j > \mathcal{N}_k$ . That is  $f_j(K) \subset \{(x, v); x \in K', v \in T_x M, r - \frac{1}{k} < |v| < r\}$  for all  $j > \mathcal{N}_k$ . Let  $k \rightarrow \infty$ , then the sequence  $\{f_j\}$  sends the compact set  $K$  to smooth strictly pseudoconvex boundary points. By assumption, the Grauert tube  $T^r M$  is not covered by the ball. The generalized Wong-Rosay theorem in [K2] implies that no subsequence of  $\{f_j\}$  could diverge compactly. A contradiction. Hence, by the tautness of  $T^r M$ , there exists a subsequence of  $\{f_j\}$  converges uniformly to some  $f \in G$  in the topology of  $G$ . Hence, some subsequence of  $\{g_j I_0\}$  converges to  $g I_0$  in the topology of  $G/I_0$ . Thus,  $G/I_0$  is compact. Since  $I$  and  $G$  have the same identity components,  $I_0$  is open in  $G$ . The compactness implies that the index of  $G/I_0$  is finite.  $\square$

For any two anti-holomorphic involutions  $\sigma$  and  $\tau$  in  $T^r M$ . We use the notation  $(M, \sigma)$  to indicate that the anti-holomorphic involution of the Grauert tube  $T^r M$  with respect to the center  $M$  is  $\sigma$ . Following the argument in Proposition 7.3 of [K2], we prove that if  $\tau$  is another anti-holomorphic involution of  $T^r M$ , then the least positive integer  $k$  such that  $(\sigma \cdot \tau)^k \in I$  is odd and in fact  $(\sigma \cdot \tau)^k = id$ .

Proposition 7.4 and Lemma 7.5 of [K2] work in the quasi-homogeneous case as well. There are at most a finite number of anti-holomorphic involutions  $\sigma_j$  in  $T^r M$ . For any center  $(N, \sigma_j)$  of  $\Omega$ , there exists  $f \in G$  such that  $f(M) = N$ . The  $Isom(M)$  is isomorphic to  $Aut(T^r M)$  if and only if there is a unique center  $(M, \sigma)$  for  $T^r M$ .

Given  $x \in M$ , we consider the homogeneous submanifold  $I_0 \cdot x$  of  $M$ .

**Lemma 4.4.** *For  $x \in K$ , the orbit  $I_0 \cdot x$  is a totally geodesic submanifold of  $M$ .*

*Proof.* We chose a slightly larger set  $\hat{K} \subset M$  such that  $K$  is relatively compact in  $\hat{K}$ . Let  $B(x, \epsilon)$  denote the intersection of  $\hat{K}$  with the geodesic ball in  $M$  centered at  $x$  of radius  $\epsilon$ . We may choose  $\epsilon$  so small such that  $B(x, \epsilon)$  is relatively compact in  $\hat{K}$  and there is no  $g \in I_0$  sending a point  $y \in B(x, \epsilon)$  to a different point  $w \in B(x, \epsilon)$ .

By the fact that  $I_0 \cdot K = M$ , it is clear that  $I_0 \cdot B(x, \epsilon) := U$  is an open subset of  $M$ , hence is a totally geodesic submanifold of  $M$  with the induced metric from  $M$ . Since there is no  $g \in I_0$  sending a point  $y \in B(x, \epsilon)$  to a different point  $w \in B(x, \epsilon)$ , we may view  $U$  as a product manifold of  $I_0$  and  $B(x, \epsilon)$ . Using the respective

induced metrics in  $U$  and  $B(x, \epsilon)$  from the Riemannian metric of  $M$ , we are able to put a Riemannian metric on  $I_0$  such that  $U$  is the product manifold of  $I_0$  and  $B(x, \epsilon)$  with the product metric.

Since  $I_0 \cdot x$  is a totally geodesic submanifold of the product manifold  $I_0 \cdot B(x, \epsilon)$  which is totally geodesic in  $M$ . Therefore,  $I_0 \cdot x$  is a totally geodesic submanifold of  $M$ .  $\square$

Let's denote  $M_x = I_0 \cdot x$ . As  $M_x$  is a totally geodesic submanifold of  $M$ , the Grauert tube  $T^r(M_x)$  is a complex submanifold of  $T^r M$ . Since  $I_0$  acts transitively on  $M_x$ , the tangent space  $T_z(T^r M_x)$  could be decomposed as, for any  $z \in T^r M_x$ ,

$$(4.5) \quad T_z(T^r M_x) = T_z(I_0 \cdot z) + T_z(T_{\pi(z)}^r M_x).$$

Following the arguments in §7 of [K-M]. We are able to construct a  $G$ -invariant strictly plurisubharmonic function

$$\psi(z) = \sum_{j=1}^k \rho(g_j(z))$$

in  $T^r M$  where  $\{g_1, \dots, g_k\} \in G$  so that  $G/G_0 = \{g_j G_0 : j = 1, \dots, k\}$  as shown in Prop. 4.3.

Let  $F_x := \psi|_{T^r M_x}$  denote the restriction of  $\psi$  to  $T^r M_x$  and  $\eta_x := \psi|_{T_{\pi(z)}^r M_x}$  the restriction of  $\psi$  to  $T_{\pi(z)}^r M_x$ . Since  $\psi$  is constant in  $I_0 \cdot z$ , by the decomposition (4.5), a critical point of  $\eta_x$  is a critical point of  $F_x$ .

As  $T^r M_x$  is a submanifold of  $T^r M$ , the restriction  $F_x$  of  $\psi$  to  $T^r M_x$  is still strictly plurisubharmonic. The real Hessian of  $F_x$  has at least  $k = \dim(I_0 \cdot x)$  positive eigenvalues and can be null on a subspace of  $T_z(T^r M_x)$  of dimension at most  $k$ . Since  $F_x$  is constant on  $T_z(I_0 \cdot z)$ , and  $\dim(I_0 \cdot z) \geq k$ , we have  $\dim T_z(I_0 \cdot z) = k$  and the real Hessian of  $\eta_x$  is positive definite on the tangent space  $T_z(T_{\pi(z)}^r M_x)$ . By the Morse theory (c.f. p85, [C-E])  $\eta_x$  has at most one critical point. Since  $\eta_x$  is proper on the fiber  $T_{\pi(z)}^r M_x$ , it follows that there is exactly one critical point of  $\eta_x$  which turns out to be the minimal point. Since  $\eta_x \cdot \sigma = \eta_x$ , the minimum of  $\eta_x$  occurs at  $\pi(z)$ .

On the other hand, the restriction  $\psi_p$  of  $\psi$  to the fiber  $T_p^r M$  is proper, therefore  $\psi_p$  has local minimums in  $T_p^r M$  which must also be critical points of  $\eta_p$ . Therefore, each  $\psi_p$  has exactly one minimal point at  $p$ . Let

$$L := \max_{p \in K} \psi(p).$$

As  $\psi$  is  $G$ -invariant,  $M = I_0 \cdot K$  and  $N = f(M)$ , we have

$$\max_{p \in M} \psi(p) = \max_{p \in K} \psi(p) = \max_{z \in N} \psi(z).$$

Let  $\mathcal{A} := \{x \in K : \psi(x) = L\}$ . Pick  $q \in \mathcal{A}$  and  $z \in N \cap T_q^r M$ . Since  $z \in N$ ,  $\psi(z) \leq L$ . On the other hand,  $z \in T_q^r M$ ,  $\psi(z) \geq \psi(q) = L$ . This can't be unless  $z = q$ . That is, every maximal point lies in the intersection of different centers;  $q \in N \cap M$  and  $q$  is a maximal point for  $\psi|_M$ .

By the quasi-homogeneity of  $M$ , the maximal points for  $\psi|_M$  are  $I_0 \cdot \mathcal{A}$  and the maximal points for  $\psi|_N$  are  $f(I_0 \cdot \mathcal{A})$ . Therefore,  $I_0 \cdot \mathcal{A} \subset f(I_0 \cdot \mathcal{A})$ . Conversely, if we start from the center  $N$ , we will obtain  $f(I_0 \cdot \mathcal{A}) \subset I_0 \cdot \mathcal{A}$ . Therefore  $\psi|_N$  and  $\psi|_M$  have the same maximal point set.

For any two centers  $M$  and  $N_j$ , there exists an  $g_j \in \text{Aut}(T^r M)$  such that  $g_j(M) = N_j$ . The above argument works for any center as well. Let  $q \in \mathcal{A}$ , then  $g_j(q)$  is a maximal point for  $\psi|_{N_j}$ . Hence,  $g_j(q) \in M, \forall j$  which implies

$$\psi(q) = \sum_{j=1}^k \rho(g_j(q)) = 0.$$

As  $\psi$  is a non-negative function and  $q$  is a maximal point for  $\psi|_M$ . We conclude that  $\psi|_M \equiv 0$ . Therefore,  $g_j(M) = M$ , for all  $g_j \in \text{Aut}(T^r M)$ .

The following main theorem is thus proved.

**Theorem 4.5.** *Let  $M$  be a quasi-homogeneous manifold. Let  $T^r M$  be the Grauert tube of radius  $r < r_{\max}(M)$  which is not covered by the ball. Then  $\text{Isom}(M) \cong \text{Aut}(T^r M)$  and  $T^r M$  has a unique center.*

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INSTITUTE OF MATHEMATICS, ACADEMIA SINICA, TAIPEI 115, TAIWAN  
*E-mail address:* `kan@math.sinica.edu.tw`